A Non-Linear Oscillator

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Abstract

In this paper, the motion of an electron is theoretically examined as it is dropped from rest through a ring of linear charge density. First, the particle is dropped close to the ring; the particle's motion is described by a 2nd order linear differential equation (DE) resulting in Simple Harmonic Motion (SHM). Next, the particle is released from any height. The electron's motion is then characterized by a 2nd order non-linear DE; this situation is also examined for the case where radiation damping is factored in. The non-linear DEs are much more difficult to solve analytically which forces us to turn to numerical methods for a solution. Since numerical methods are not part of many physics undergraduates' curriculum, a simple method is illustrated on how to employ Mathematica's `NDSolve` (numerical DE solver) function to find graphical representations of the electron's amplitude and velocity as functions of time.

I. Introduction

Many physical systems are described mathematically by nonlinear DEs. Undergraduate students of physics are often limited to studying simplified versions of these systems because of their lack of experience in solving these types of equations; analytically or numerically.

An example of such a simplified system is the "simple" pendulum. In its primitive form, the equation of motion for a simple pendulum is,

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0\,, \qquad (1)$$

where ω_0, represents the system's angular frequency and θ the system's amplitude. This is a 2nd order non-linear differential equation in θ. In undergraduate mechanics courses, (1) is simplified by "linearizing" it using a small angle approximation. Here, only small oscillations of the pendulum about its equilibrium position are considered so that for small θ, $\sin\theta \approx \theta$. This reduces (1) to a linear DE in θ, a much more familiar, and analytically solvable, problem.

The difficult task of solving nonlinear DEs analytically usually forces one to find numerical solutions to the problem; using, for example, finite difference methods. There are numerous software tools which dramatically simplify this daunting task; enter Mathematica. Mathematica's `NDSolve` function allows one to find a numerical solution to a given nonlinear DE (or a linear DE for that matter) by inputting the DE and its corresponding initial and/or boundary values. The numerical results can then be plotted with Mathematica. These plots then provide a visual for one to analyze and gather further insight into the behavior of the real world system.

In this paper we will examine the motion of an electron as it moves under the influence of a ring of positive linear charge density. It turns out that the electron's equation of motion, when it is constrained to move along the z-axis, is given by a second order nonlinear DE, resulting in oscillatory motion. This equation is first linearized, similar to the method described above, to obtain a simplified expression for the electron's amplitude and velocity as a function of time.

Mathematica's `NDSolve` function is then used to find a numerical solution for the electron's velocity and amplitude for the nonlinear form of the electron's equation of motion; these values are then plotted and analyzed. As a second example, a radiation damping force on the electron is considered. The resulting equation of motion is another nonlinear DE. Numerical solutions for the electron's velocity and amplitude are found, plotted, and then analyzed.

II. Physical Situation and Derivation of the Electron's Equation of Motion

A. Physical Situation

The situation is illustrated in Fig 1. An electron is at rest at point P above a ring of linear charge density λ; the point P is described by $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ relative to the source charge dq, with $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\vec{r}' = R\cos\phi + R\sin\phi$ being the positions of P and dq respectively, relative to the origin. A differential element of the ring, dq, produces a differential electric field at the point P given by:

$$dE = \frac{dq}{4\pi\varepsilon_0}\frac{\vec{\mathscr{r}}}{\vec{\mathscr{r}}^3}. \tag{2}$$

Plugging the value of $\vec{\mathscr{r}}$ into (2) and integrating over $\phi : 0 \rightarrow 2\pi$ we get the following expression for the electric field produced by the ring,

$$\vec{E} = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{(x - R\cos\phi)\hat{x} + (y - R\sin\phi)\hat{y} + z\hat{z}}{[(x - R\cos\phi)^2 + (y - R\sin\phi)^2 + z^2]^{3/2}}d\phi, \tag{3}$$

where I used $dq = \lambda R d\phi$. (3) is a complicated elliptic-type integral which is too difficult to handle here in our treatment. Thus, we constrain the electron to move along the z-axis (i.e. parallel to the plane of the ring). Putting x = 0, y = 0 in (3) we have,

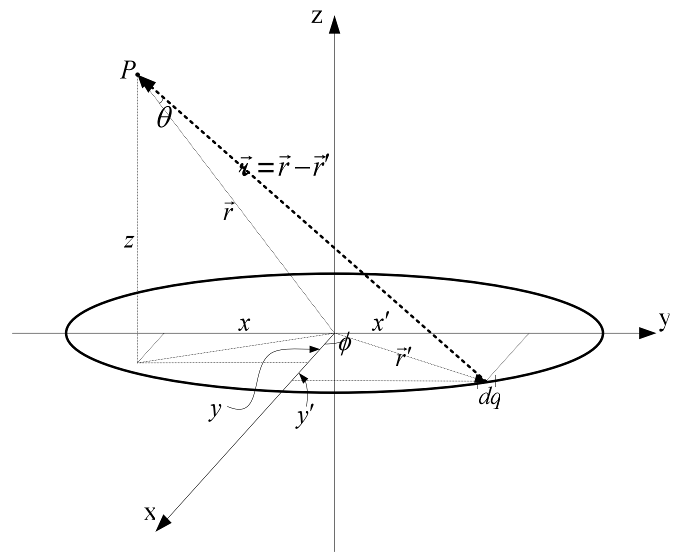


Figure 1. If an electron is placed at point P, its position is described by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ relative to the origin, and by $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ relative to *dq* on the ring with total charge *q*. $\vec{r}'$ is the position of *dq* relative to the origin.

$$\vec{E} = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{-R\cos\phi\hat{x} - R\sin\phi\hat{y} + z\hat{z}}{[R^2 + z^2]^{3/2}}d\phi. \tag{4}$$

The integrals for E_x and E_y are both zero, as expected, so that the net E-field points in the positive $\hat{z}$ direction and is given by the well-known result[1],

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2 + z^2)^{3/2}}\hat{z}, \tag{5}$$

where $2\pi R\lambda = Q$, Q being the total charge of the ring.

B. Electron's Equation of Motion

In the presence of the ring's electric field, the electron experiences an attractive Lorentz force given by, $\vec{F} = e\vec{E}$, where *e* is the electron's charge. The particle's equation of motion is then derived from Newton's 2nd law of motion (neglecting the gravitational influence):

[1] Fundamentals of Physics: Extended 5th Edition, (Halliday, Resnick, Walker), p. 562

$$\ddot{z} + \frac{e}{4\pi\varepsilon_0 m}\frac{Qz}{(R^2+z^2)^{3/2}} = 0 . \qquad (6)$$

This is a 2nd order non-linear differential equation; its solution describes the electron's amplitude as a function of time.

For ease of analysis, (6) is made dimensionless by making the scaling substitutions $\xi = z/R$ and $\tau = \omega_0 t$ so that (6) becomes,

$$\ddot{\xi} + \frac{\xi}{(\xi^2+1)^{3/2}} = 0 , \qquad (7)$$

where,

$$\omega_0{}^2 = \frac{eQ}{4\pi\varepsilon_0 m R^3} . \qquad (8)$$

The solution to (7), $\xi(\tau)$, gives the amplitude of the electron when is it released from any height ξ_0 along z-axis.

Electromagnetic theory predicts that an accelerating charge radiates energy. Furthermore it predicts that this loss of energy results in a "radiation reaction" where the radiation produces a counteractive force on the electron as it releases its energy. The radiation force is given by[2],

$$F_{rad} = m\gamma\dddot{z} , \qquad (9)$$

where,

$$\tau \equiv \frac{\mu_0 e^2}{6\pi mc} . \qquad (10)$$

Adding the radiation reaction force (9) to the left hand side of (6), using the above-mentioned scaling factors ξ and τ, and a new scaling factor $\gamma = \tau'\omega^2$, we get the following dimensionless DE,

$$\ddot{\xi} + \omega\tau'\dot{\xi} + \frac{\xi}{(\xi^2+1)^{3/2}} = 0 . \qquad (11)$$

This is our second 2nd order nonlinear DE; it characterizes the electrons motion when radiation damping is factored in.

[2] Introduction to Electrodynamics: 3rd Edition, David J. Griffiths p. 467-468

C. Releasing the Electron Close to the Ring: Linearizing the DE

Consider the limiting case where z << R (i.e. electron is dropped close to the plane of the ring). We can then neglect the ξ^2 term in the denominator of (7); it reduces to:

$$\ddot{\xi} + \xi = 0 . \qquad (12)$$

This is a 2nd order linear differential equation in ξ which is well known as the DE that results in SHM[3].

The electron thus oscillates between two classical turning points as dictated by SHM theory. This motion results from two competing forces. When the particle is released, the ring attracts it governed by Lorentz force law. The electron gains momentum as it accelerates in the $-\hat{z}$ direction; it then reaches a point below the ring where the attractive rings field finally slows the electron to a halt and causes it to retreat in the $+\hat{z}$ direction where it ultimately stops and heads back in the $-\hat{z}$ direction. The resulting motion is oscillatory where the stopping amplitudes of the electron both below and above the ring are the classical turning points.

The challenge now is to solve (6), (11), and (12), analytically or numerically for $\xi(\tau)$.

III. Solutions to the Linear and Non-Linear Equations with and without Radiation Damping

A. Linear Case, no Radiation Damping

The analytic solution to (9) can be found in any undergraduate physics text[4]. It characterizes a system that undergoes SHM. Taking the initial amplitude and initial velocity to be ξ_0 and 0 respectively, we find the solutions,

$$\xi(\tau) = \xi_0 \cos(\tau) , \quad \text{(Position)} \qquad (13)$$

[3] Fundamentals of Physics: Extended 5th Edition, (Halliday, Resnick, Walker), p. 375

[4] See for example Fundamentals of Physics 5th Edition (Halliday, Resnick, Walker), p. 374

$$\xi'(\tau) = -\xi_0 \sin(\tau). \text{ (Velocity)} \qquad (14)$$

Even though it is not necessary here, we can just as easily find a numerical solution to this equation using Mathematica's `NDSolve` function (see appendix A for Mathematica code). (13) and (14) are plotted below where we take $\xi_0 = 0.5$.

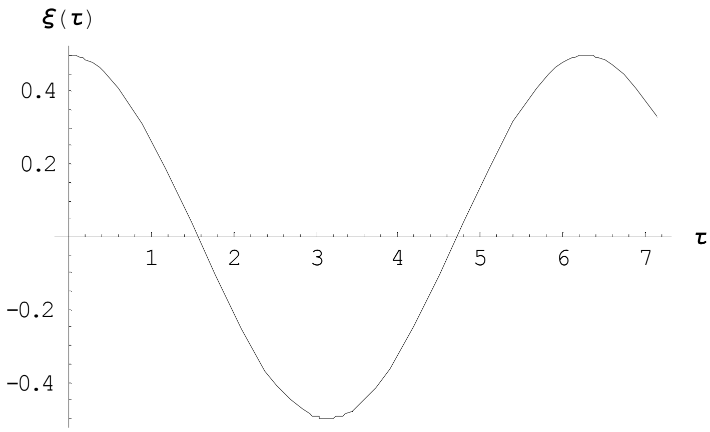
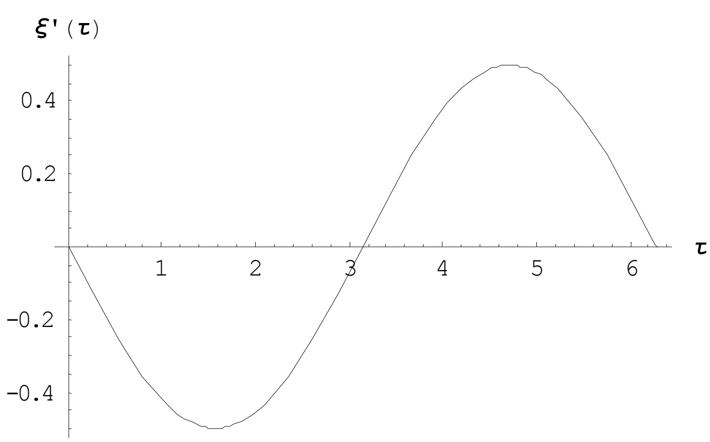
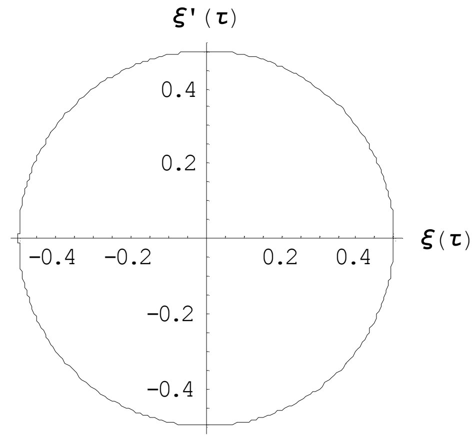


Figure 2. (a) (above) The scaled position of the electron as a function of τ, for the linear case (SHM), with $\xi_0 = 0.5$. (b) (below) The scaled velocity of the electron as a function of τ. Both plots are sinusoidal as expected for SHM. (c) (top right) Phase plot of $\xi(\tau)$ vs. $\xi'(\tau)$.

We can see from plots 2(a) and 2(b) that the classical turning points for this system are at $\xi(\tau) = \pm\xi_0$.





B. Non-Linear Equation

1. No Radiation Damping

We don't have the luxury of an analytic solution to (7) as we did in the linear case (12). Thus we use Mathematica's `NDSolve` function to numerically solve (7) for $\xi(\tau)$ and $\xi'(\tau)$ (see appendix B for Mathematica code). These quantities are plotted in figure 3 below, along with the system's phase plot; here I took $\xi_0 = 2.0$.

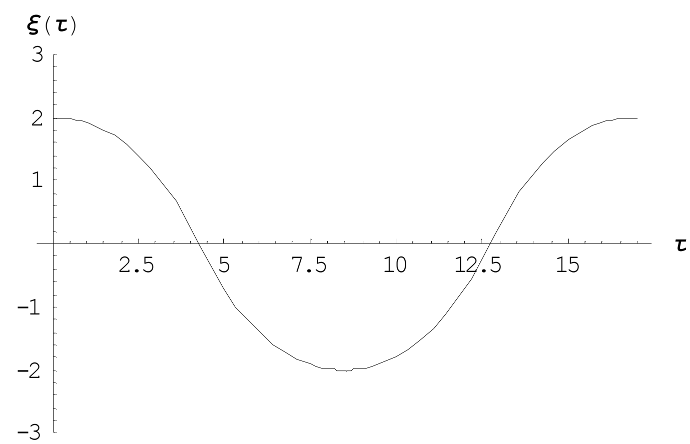
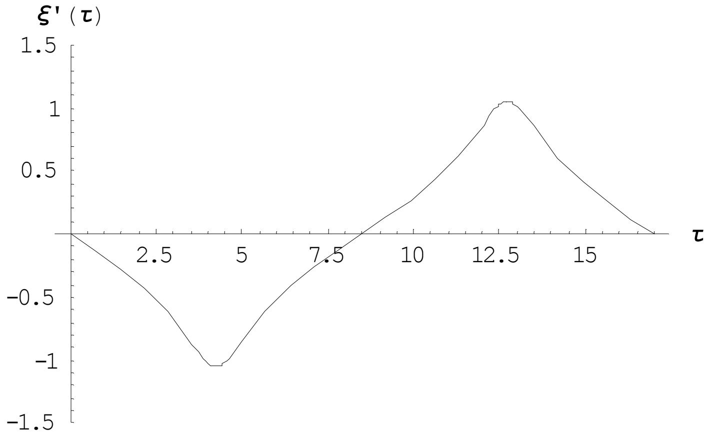
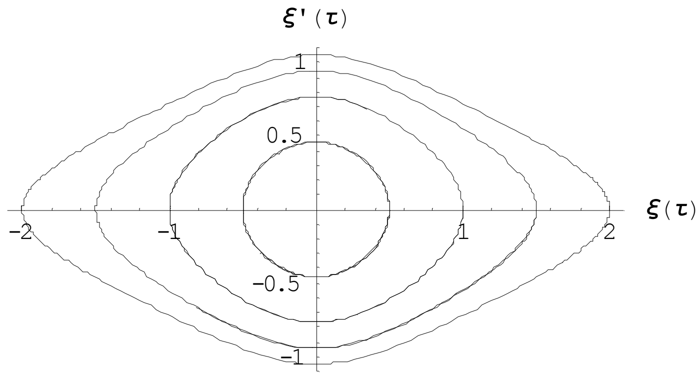


Figure 3. (a) (above) The amplitude of the electron as a function of τ for the nonlinear case, with $\xi_0 = 2$. (b) (below) The velocity of the electron as a function of τ. (c) (below b) Phase plot $\xi(\tau)$ vs. $\xi'(\tau)$.





2. With Radiation Damping

Here we seek a numerical solution to (11); this will illustrate how the electron's amplitude and velocity are affected by the radiation damping. We expect that the electron will still oscillate; however, its amplitude will eventually die out and go to zero due to the electron's loss of energy as τ increases. These quantities are plotted in figure 3 below (see appendix C for Mathematica code), along with the system's phase plot; here I took $\xi_0 = 2.0$. The plot in Figure 4(a) validates our suspicion about the damping of the electron's amplitude as it shows that its amplitude is indeed oscillatory with decreasing amplitude as τ increases.

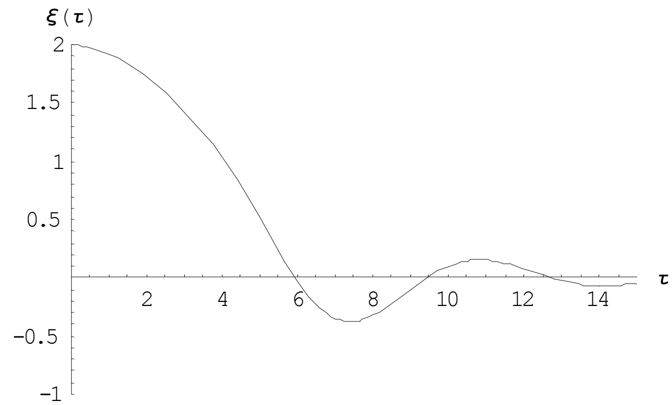
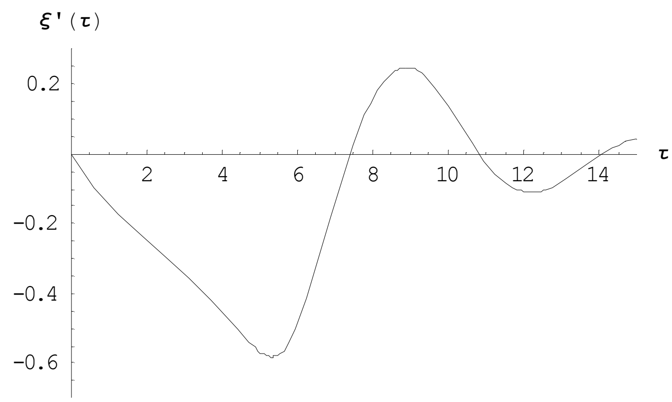
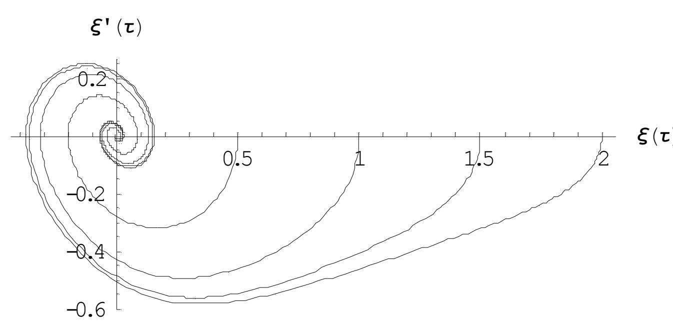


Figure 4. (a) (above) Amplitude of the electron as a function of τ when radiation damping is factored in to the nonlinear equation. Here $\xi_0 = 2$ (b) (below) Velocity of the electron as a function of τ. (c) (below b) Phase plot $\xi(\tau)$ vs. $\xi'(\tau)$.





IV. Discussion

In this paper we have successfully solved two nonlinear DEs using Mathematica's `NDSolve` function. The DEs characterized the motion of an electron as it oscillates through a ring of linear charge density. While we did not even attempt to find a closed-form solution of either of the equations, it was sufficient to examine plots of the electron's amplitude, produced numerically, to illustrate interesting electron trajectories under the varying condtions. The method presented here could easily be implemented in an undergraduate or high school physics course as a comparison of the real-world (nonlinear) and simplified (linear) cases.

The problem was made complicated in one sense by solving the nonlinear DEs, instead of linearized versions of them, but there were simplifications in other areas of the problem. For example, the charge on the ring was taken to be linear. The problem could have been complicated more, and possibly made it even more interesting, by giving the charge density some kind of time or angular dependence (i.e. $\lambda(t)$ or $\lambda(\phi)$). Furthermore, the electron was constrained to move along the z-axis. If the electron is dropped off the z-axis, the problem is greatly complicated since the integral in (3) is no longer trivial. However it would be very interesting to see if the electron would remain within the range of the ring (i.e. electron travel freely within imaginary circular cylinder $-R \leq x, y \leq R$). Both of these modifications or enhancements are valid avenues for further study.

V. Acknowledgements

I would like to thank Dr. Tahsiri for this opportunity to study this interesting problem.